Space Charge Tune Shift in PIC

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1) Introduction

Parameters from the PIC/REMEX revised paper

		Pic 1	PIC 2
Cell lengths	cm	19	19
Momentum	MeV/c	100	100
Muons/bunch		10^{11}	10^{11}
Absorber thick	mm	6.4	1.6
Absorber Mat		Be	Be
Trans RMS emit	mm mrad	600	30
Sigma(theta)	Mrad	200	200
Sigma(r)	mm	3	0.15
$\beta_{\rm Beam}$	mm	15	0.75
ϵ_o	mm mrad	118	6.0
RMS dp/p	%	3	3
Sigma(z)	cm	0.5	0.5
Long RMS emittance	cm	0.015	0.015

The blue numbers differ from the original paper

The red numbers are calculated on right

$$\beta_\perp = \frac{\sigma_{x,y}}{\sigma_{\theta x,\theta y}}$$

$$= 15 \rightarrow \quad (0.75 \text{ mm})$$

material	T ^{o}K	density kg/m^3	dE/dx MeV/m	L_R m	C_o 10^{-4}
Liquid H_2	20	71	28.7	8.65	38
Liquid He	4	125	24.2	7.55	51
LiH	300	820	159	0.971	61
Li	300	530	87.5	1.55	69
Be	300	1850	295	0.353	89
Al	300	2700	436	0.089	248

$$\epsilon_o = \frac{\beta_\perp}{\beta_v} C_{\rm Be} \frac{dE/dx(min)}{dE/dx(p)}$$

$$\approx 118 \rightarrow 6.0 \quad (10^{-6}\text{m})$$

Introduction Continued

- There has been considerable confusion about the space charge tune shift in the proposed PIC lattices
- This confusion arises from the special case of PIC operation
  - In a PIC lattice there is a deliberate miss-match between the beam beta ($\beta_{\text{beam}} = \sigma_x/\sigma_\theta x$) and the latice beta ($\beta_{\text{lattice}}$)
  - The lattice operates on a half integer resonance: $\nu = 0.5$
- The confusion arises because when $\beta_{\text{beam}} \neq \beta_{\text{lattice}}$ then there are two different phase advances and thus two different tunes and tune shifts:

$$\chi_{\text{lattice}} = \int \frac{ds}{\beta_{\text{lattice}}} \qquad \chi_{\text{beam}} = \int \frac{ds}{\beta_{\text{beam}}}$$

$$\nu_{\text{lattice}} = \int \frac{ds}{2\pi\ \beta_{\text{lattice}}} \qquad \nu_{\text{beam}} = \int \frac{ds}{2\pi\ \beta_{\text{beam}}}$$

- In the following we will re-derive the space charge tune shifts for each definition and examine the constraints on each
- In the following $\beta_v = v/c$. The emittance $\epsilon_\perp$ is the normalized rms value, so that for an upright ellipse:

$$\epsilon_\perp = \beta_v \gamma\ \sigma_x\ \sigma_\theta$$

2) Space Charge Force

The defocus radial force $$F(r) = \left(\frac{2mc^2\ r_\mu}{\gamma^2}\right)\ \frac{(dN(r)/ds)}{r}$$

where $dN(r)/ds$ is the total line charge density inside a radius r

$$(dN(r)/ds) = \int_o^r 2\pi r\ (dn(r)/ds)\ dr$$

for a flat distribution up to a radius a

$$(dN(r)/ds) = (dn(o)/ds)\ \pi\ r^2 = (dN(\infty)/ds)\ \frac{r^2}{a^2}$$

for a Gaussian distribution in r

$$(dN(r)/ds) = (dN(\infty)/ds)\ \left(e^{\frac{r^2}{2\sigma^2}}\right)\ \frac{r^2}{2\sigma^2}$$

For small r and flat (as given in SY Lee p109)

$$F(r) = \left(\frac{2mc^2\ r_\mu}{\gamma^2}\right)\ \left(\frac{(dN(\infty)/ds)}{a^2}\right)\ r$$

For small r and gaussian

$$F(r) = \left(\frac{2mc^2\ r_\mu}{\gamma^2}\right)\left(\frac{(dN(\infty)/ds)}{2\sigma^2}\right)\ r$$

3) Defocus Strength K

$F(r)$ introduces a 'quadrupole' like defocus, in both x and y, of strength $K_{sc}(r)$:

$$K_{sc}(r) \quad = \quad \frac{(dN(\infty)/ds)\ r_\mu}{\sigma^2\ \beta_v^2\ \gamma^3}$$

Using

$$\sigma^2 \quad = \quad \frac{\epsilon_\perp\ \beta_{\rm beam}}{\beta_v \gamma}$$

$$K_{sc}(r) \quad = \quad \frac{(dN(\infty)/ds)\ r_\mu}{\epsilon_\perp\ \beta_{\rm beam}\ \beta_v\ \gamma^2}$$

Note that the $\beta_{\rm beam}$ must indeed be the beam parameter, not the lattice, since it sets the beam dimension

For a bunch with Gaussian longtudinal shape

$$K_{sc}(r) \quad = \quad \left(\frac{N_\mu}{\sqrt{2\pi}\ \sigma_z}\right)\ \left(\frac{r_\mu}{\epsilon_\perp\ \beta_v\ \gamma^2}\right)\ \left(\frac{1}{\beta_{\rm beam}}\right)$$

4) Resulting Tune Shifts (eg SY Lee p92)

$$\Delta\nu_{\rm lattice} \;=\; \frac{1}{4\pi}\; \int_o^L \beta_{\rm lattice}\; K_{sc} ds$$

Which came from Courant & Schnsider and is on solid ground

By analogy, but on weaker ground, we can define

$$\Delta\nu_{\rm beam}(o) \;=\; \frac{1}{4\pi}\; \int_o^L \beta_{\rm beam}\; K_{sc} ds$$

$$\Delta\nu_{\rm beam}(o) \;=\; \left(\frac{N_\mu}{\sqrt{2\pi}\;\sigma_z}\right)\; \frac{r_\mu}{4\pi\;\epsilon_\perp\;\beta_v\gamma^2} \left(\oint \frac{\beta_{\rm beam}}{\beta_{\rm beam}} ds\right) \;=\; \left(\frac{N_\mu\; r_\mu}{\sqrt{2\pi}\;\sigma_z\;4\pi\;\epsilon_\perp\;\beta_v\gamma^2}\right)\; L_{\rm cell}$$

This is true INDEPENDENT of $\beta_\perp$ and proportional to $1/\epsilon_\perp$. However, there remains a question as to whether the derivation of $\Delta\nu_{\rm beam}$ is correct so we should look at $\Delta\nu_{\rm lattice}$

$$\Delta\nu_{\rm lattice}(o) \;=\; \left(\frac{N_\mu\; r_\mu}{\sqrt{2\pi}\;\sigma_z\;4\pi\;\epsilon_\perp\;\beta_v\gamma^2}\right)\; \left(\oint \frac{\beta_{\rm lattice}}{\beta_{\rm beam}} ds\right)$$

This is not independdent of s because the integration includes the s dependent term $(\beta_{\rm lattice}/\beta_{\rm beam})$. Note that this term, in the PIC case is greater than one and is rising as $\beta_{\rm beam}$falls as a result of the falling $\epsilon_\perp$. So $\Delta\nu_{\rm lattice}$ is NOT independent of $\beta_\perp$, and is NOT indpendent of $\epsilon_\perp$

5) PIC Constrains on $\Delta\nu_{\rm lattice}$

Before a cell, the ellipse should be upright with

$$\sigma_x(1) \;=\; \frac{\epsilon_\perp}{\beta_v\gamma\ \sigma_\theta}$$

If there is an error in $\nu_{\rm lattice}$ of $\Delta\nu_{\rm lattice}$,
The projected size of the beam after a cell:

$$\sigma_x(2) \;=\; \rightarrow\ \sigma_\theta\ \beta_{\rm lattice}\ \sin(\Delta\nu_{\rm lattice})$$

Our requirement is that $\sigma_x(2) << \sigma_x(1)$:

$$\sigma_\theta\ \beta_{\rm lattice}\ \sin(\Delta\nu_{\rm lattice}) \;<<\; \frac{\epsilon_\perp}{\beta_v\gamma\ \sigma_\theta}$$

$$\Delta\nu_{\rm lattice} \;<<\; \frac{\epsilon_\perp}{\beta_v\gamma\ \sigma_\theta^2\ \beta_{\rm lattice}}$$

The denominator is a constant, so
as $\epsilon_\perp$ falls with cooling, the requirement gets ever tighter

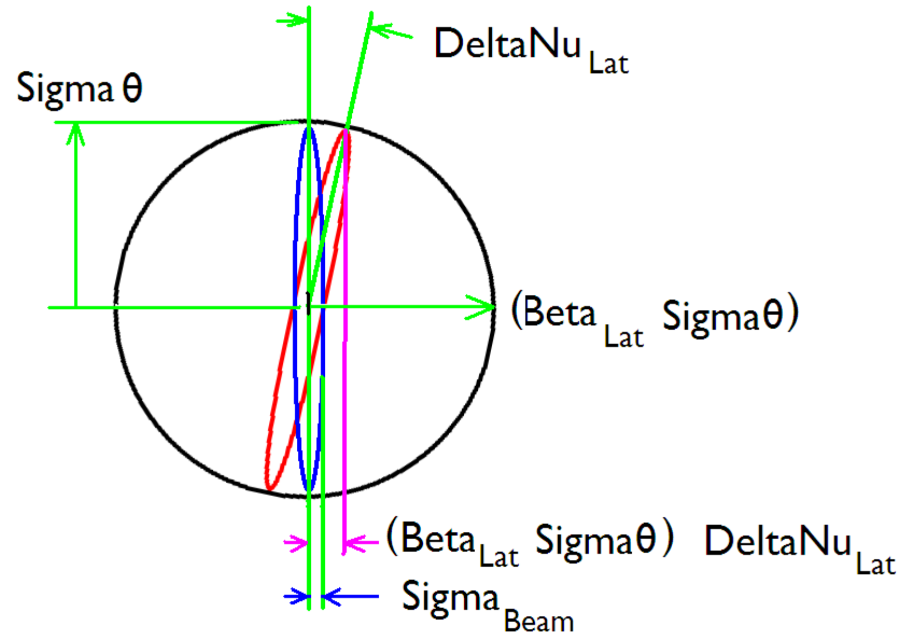

Figure is for uniform focusing, as in a solenoid

Numerically we must assure that the increase in σ is less than the decrement in σ from the cooling. Since, at the end of PIC, this decrement is only about 0.3%, the constraint on $\Delta\nu_{\rm lattice}$ will be very tight

6) Conclusion

- The beam defined tune shift is inpependent of the lattice β_{lattice} and inversely proportional to emittance
- But some questions remain about the derivation of this parameter in PIC's unmatched condition
- The more conventioal, lattice defined, tune shift $\Delta\nu_{\mathrm{lattice}}$ is not independent of the $\beta_{\perp}$s
  and must be computed for any particular lattice
- $\Delta\nu_{\mathrm{lattice}}$ also rises as the emittance falls, though not as fast, or in such a simple way as $\Delta\nu_{\mathrm{beam}}$
- The PIC constraint on the value of $\Delta\nu_{\mathrm{lattice}}$ is much tighter than that for $\Delta\nu_{\mathrm{beam}}$ in a conventioal matched lattice, and rises linearly as the emittance falls

- I hope to give numerical examples later